IN PLACE DISCRETE FOURIER TRANSFORM OF REAL DATA

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1. Discrete Fourier Transform

Notation 1.1. By small thick letters we will denote $d-$tuples of integers as $\mathbf{x} = (x_1, ..., x_d)$. The relations $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{x} < \mathbf{y}$ denote $x_j \leq y_j$ and $x_j < y_j$ for all $j = 0, .., d$, respectively. Finite $d-$dimensional arrays of size $n_1 \times n_2 \times ... \times n_d$ will be denoted by capital letters as $\mathbf{A} = \{a_\mathbf{x}\}_{0 \leq \mathbf{x} < \mathbf{n}}$.

Definition 1.2. Let $\mathbf{A} = \{a_\mathbf{x}\}_{0 \leq \mathbf{x} < \mathbf{n}}$ be $d-$dimensional array of complex numbers $a_\mathbf{x}$. Discrete Fourier transform (DFT) of $\mathbf{A}$ is $\mathbf{C} = \{c_\mathbf{k}\}_{0 \leq \mathbf{k} < \mathbf{n}}$, where

$$c_\mathbf{k} = \frac{1}{\sqrt{n_1 n_2 ... n_d}} \sum_{0 \leq \mathbf{x} < \mathbf{n}} a_\mathbf{x} \exp\left(2\pi i \left(\frac{x_1 k_1}{n_1} + \frac{x_2 k_2}{n_2}, ... + \frac{x_d k_d}{n_d}\right)\right).$$

Inverse operation to DFT is

$$a_\mathbf{x} = \frac{1}{\sqrt{n_1 n_2 ... n_d}} \sum_{0 \leq \mathbf{k} < \mathbf{n}} c_\mathbf{k} \exp\left(-2\pi i \left(\frac{x_1 k_1}{n_1} + \frac{x_2 k_2}{n_2}, ... + \frac{x_d k_d}{n_d}\right)\right).$$

Software implementations of DFT use its property, that it can be separated (e.g. rowwise, then columnwise etc.) to sequence of $1-$dimensional DFTs of sizes $n_1, n_2, ..., n_d$ and the $1-$dimensional transforms can be implemented effectively e.g. for n equal to a power of 2 by fast Fourier transform (FFT). The FFT algorithms can work in place, i.e. the transform implementation based on $1-$dimensional FFT replaces in memory the values of $a_\mathbf{x}$ by the values $c_\mathbf{x}$.

DFT of complex conjugate array $\mathbf{A}^* = \{a^*_\mathbf{x}\}_{0 \leq \mathbf{x} < \mathbf{n}}$ is reflected conjugated array $\overline{\mathbf{C}^*} = \left\{c^*_{\mathbf{n}-\mathbf{k}}\right\}_{0 \leq \mathbf{k} < \mathbf{n}}$. Two corollaries follow:

i) Let the complex array $\mathbf{D} = \{d_\mathbf{k}\}_{0 \leq \mathbf{k} < \mathbf{n}}$ be the DFT of real array $\mathbf{B} = \{b_\mathbf{x}\}_{0 \leq \mathbf{x} < \mathbf{n}}$. Then the relation $d_\mathbf{k} = d^*_{\mathbf{n}-\mathbf{k}} = d^*_{(n_1-k_1, ..., n_d-k_d)}$ holds for all $\mathbf{k}$, $0 \leq \mathbf{k} < \mathbf{n}$. So if we know the coefficients with indices $k_1 \leq \frac{n_1}{2}$, we can calculate all the coefficients $d_\mathbf{k}$, $0 \leq \mathbf{k} < \mathbf{n}$.

ii) Let $\mathbf{C}$ be DFT of $\mathbf{A}$. DFT of real array $Re\mathbf{A}$ formed by the real parts of the field $\mathbf{A}$ is $\frac{1}{2}\left(\mathbf{C} + \overline{\mathbf{C}^*}\right)$, DFT of real array $Im\mathbf{A}$ formed by the imaginary parts of the field $\mathbf{A}$ is $\frac{1}{2i}\left(\mathbf{C} - \overline{\mathbf{C}^*}\right)$.

2. DFT of Real Array

Let $\mathbf{B} = \{b_\mathbf{x}\}_{0 \leq \mathbf{x} < (2n_1, ..., n_d)}$ be finite $d-$dimensional array of size $2n_1 \times n_2 \times ... \times n_d$ of real numbers such, that $a_{(x_1, ..., x_d)} = b_{(2x_1, ..., x_d)} + ib_{(2x_1+1, ..., x_d)}$ for the complex array $\mathbf{A} = \{a_\mathbf{x}\}_{0 \leq \mathbf{x} < \mathbf{n}}$ and let $\mathbf{C} = \{c_\mathbf{k}\}_{0 \leq \mathbf{k} < \mathbf{n}}$ be its DFT. Let $\mathbf{D} = \{d_\mathbf{k}\}_{0 \leq \mathbf{k} < (2n_1, ..., n_d)}$ be DFT of $\mathbf{B}$. From the definition of DFT of $\mathbf{D}$ and the

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corollary ii) it follows, that $d_{\mathbf{k}} = \frac{1}{2}\left(c_{\mathbf{k}} + c^*_{\mathbf{n}-\mathbf{k}}\right) + \frac{1}{2i}\left(c_{\mathbf{k}} - c^*_{\mathbf{n}-\mathbf{k}}\right)\exp\left(2\pi i \frac{k_1}{n_1}\right)$ for $0 \leq \mathbf{k} \leq (n_1, n_2 - 1, ..., n_d - 1)$. Half of the values of $d_{(n_1,k_2,...,k_d)}$ where $(k_2, ..., k_d) \neq (n_2 - k_2, ..., n_d - k_d)$ can be stored in place of values of $d_{(0,n_2-k_2,...,n_d-k_d)}$. Values of $d_{(n_1,k_2,...,k_d)}$ where $(k_2, ..., k_d) = (n_2 - k_2, ..., n_d - k_d)$ are real and can be stored in place of imaginary part of $d_{(0,k_2,...,k_d)}$. The values of coefficients $d_{\mathbf{k}}$ for $n_1 < k_1 < 2n_1$ can be calculated by corollary i), if needed.

We can calculate an inverse transform in place in a similar way. Calculation DFT of cyclic convolution of two real $d-$dimensional arrays in frequency domain is also straightforward.

So we are able to calculate DFT of $d-$dimensional array of real numbers with even number of elements in first dimension in place using only half of the memory needed for calculation of DFT of array of complex numbers with zero imaginary part.