Recursive Gaussian after Young and de Vliet

by Jiří Janáček

Gaussian

1D

$$y[n] = (g \otimes x)[n] \equiv \sum_{k=-\infty}^{\infty} g[k] \cdot x[k-n]$$

$$g[k] = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{k^2}{2\sigma^2}}$$

2D

$$y[m,n] = (g \otimes x)[m,n] \equiv \sum_{j,k=-\infty}^{\infty} g[j,k] \cdot x[j-m, k-n] =$$

$$g[j,k] = \frac{1}{\left(\sqrt{2\pi}\sigma\right)^2} e^{-\frac{j^2+k^2}{2\sigma^2}} = g[j] \cdot g[k]$$ is separable (and rotationaly symmetric)

$$\sum_{j,k=-\infty}^{\infty} g[j,k] \cdot x[j-m, k-n] = \sum_{j=-\infty}^{\infty} g[j] \cdot \sum_{k=-\infty}^{\infty} g[k] \cdot x[j-m, k-n]$$

3D

$$y[l,m,n] = (g \otimes x)[l,m,n] \equiv \sum_{i,j,k=-\infty}^{\infty} g[i,j,k] \cdot x[i-l, j-m, k-n]$$

$$g[i,j,k] = \frac{1}{\left(\sqrt{2\pi}\sigma\right)^3} e^{-\frac{i^2+j^2+k^2}{2\sigma^2}} = g[i] \cdot g[j] \cdot g[k]$$ is separable (and rotationaly symmetric)

$$\sum_{i,j,k=-\infty}^{\infty} g[i,j,k] \cdot x[i-l, j-m, k-n] = \sum_{i=-\infty}^{\infty} g[i] \cdot \sum_{j=-\infty}^{\infty} g[j] \cdot \sum_{k=-\infty}^{\infty} g[k] \cdot x[i-l, j-m, k-n]$$

thus it is sufficient to do:

Approximate evaluation in 1D, direct:

$$y[n] \cong \sum_{k=-N}^{N} g[k] \cdot x[k-n]$$

for $N = \lceil 5\sigma \rceil$ je $g[N] \le 3.7 \cdot 10^{-6} \cdot g[0]$

$2N+1$ multiplications, $2N$ additions per point.

Approximate evaluation in 1D, for big σ, recursive:

Z-transform of recursive filter:

$$\sum_{j=0}^{M} y[n-j]\cdot a[j] = \sum_{k=0}^{N} x[n-k]\cdot b[k]$$

$$a[0]=1$$

$$y[n] = \sum_{k=0}^{N} x[n-k]\cdot b[k] - \sum_{j=1}^{M} y[n-j]\cdot a[j]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n}$$

$$Y(z)\cdot \sum_{j=0}^{M} z^{-j}a[j] = X(z)\cdot \sum_{k=0}^{N} z^{-k}b[k]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{j=0}^{M} z^{-j}a[j]}{\sum_{k=0}^{N} z^{-k}b[k]}$$

$$H(z) = \frac{\prod_{j=0}^{M}\left(1-q[j]z^{-j}\right)}{\prod_{k=0}^{N}\left(1-p[k]z^{-k}\right)}$$, q are poles, p are roots.

Rational approximation of Gaussian:

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}} = \frac{1}{a[0]+a[2]t^2+a[4]t^4+a[6]t^6} + \varepsilon(t)$$

$a[0]=2.490895$, $a[2]=1.466003$,
$a[4]=-0.024393$, $a[6]=0.178257$, $|\varepsilon(t)| < 2.7\cdot 10^{-3}$

$e^{-\frac{\sigma^2\omega^2}{2}}$ is Fourrier transform of Gaussian,
replace σ by q, $j\omega$ by s

$$\frac{a[0]}{a[0]-a[2]q^2s^2+a[4]q^4s^4-a[6]q^6s^6}$$

$$\frac{a[0]}{(qs+m[0])(qs+m[1]+jm[2])(qs+m[1]-jm[2])}\cdot$$

$$\cdot\frac{1}{(qs-m[0])(qs-m[1]-jm[2])(qs-m[1]+jm[2])}$$

$m[0]=1.16680$, $m[1]=1.10783$, $m[2]=1.40586$

backward difference $s=1-z^{-1}$, direct difference $s=1-z$:

$$H_+(z) = \frac{1}{b[0]+b[1]z^{-1}+b[2]z^{-2}+b[3]z^{-3}}$$

$$H_{-}(z) = \frac{B}{b[0] + b[1]z + b[2]z + b[3]z}$$

So, the calculation of the 1D recursive Gaussian is:

$$w[n] = \frac{1}{b[0]}\left(x[n] - b[1]w[n-1] - b[2]w[n-2] - b[3]w[n-3]\right)$$

$$y[n] = \frac{1}{b[0]}\left(Bw[n] - b[1]y[n+1] - b[2]y[n+2] - b[3]y[n+3]\right)$$

where:

$$b[0] = 1,\ scale = \left(m[0] + q\right)\left(m[1]^2 + m[2]^2 + 2m[1]q + q^2\right),$$

$$b[1] = -\frac{q}{scale}\left(2m[0]m[1] + m[1]^2 + m[2]^2 + \left(m[0] + 4m[1]q + q^2\right)\right),$$

$$b[2] = \frac{q^2}{scale}\left(m[0] + 2m[1] + 3q\right),$$

$$b[3] = -\frac{q^3}{scale}$$

$$q(\sigma) = 1.31564 \cdot \left(\sqrt{1 + 0.490811 \cdot \sigma^2} - 1\right)$$

6 multiplications, 6 additions per point.

For correction of boundary defects see Triggs and Sdika.

Gabor filter

$$w[n] = \frac{1}{b[0]}\left(x[n] - b[1]e^{j\Omega}w[n-1] - b[2]e^{2j\Omega}w[n-2] - b[3]e^{3j\Omega}w[n-3]\right)$$

$$y[n] = \frac{1}{b[0]}\left(Bw[n] - b[1]e^{-j\Omega}y[n+1] - b[2]e^{-2j\Omega}y[n+2] - b[3]e^{-3j\Omega}y[n+3]\right)$$

Reference:

Young I.T., van Vliet L.J.: Recursive Gabor filtering, IEEE Trans. Signal Processing 50 (2002), 2798-2805.

Triggs B., Sdika M.: Boundary conditions for Young-van Vliet recursive filtering, IEEE Trans. Signal Processing 54 (2006), 2365-2367.