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MINIMIZATION OF TOTAL VARIATION IN IMAGE ANALYSIS

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Abstract: A solution of various problems in image analysis using concurrent minimization of total variation and $\mathbf{L}^p$ loss function is presented. The minimization is achieved by a steepest descend method using graph cut minimization in each step. Regularization of noisy image and registration of microscopic images of physical sections are demonstrated.

1 Introduction

Various tasks of image analysis and computer vision can be solved by optimization of suitable cost functions. An example is image filtration with cost function penalizing functions roughness and distance from the original image. Variety of deterministic and stochastic approaches was applied to the optimization and it appeared that the stochastic methods were too slow and the deterministic methods yielded only local optima. The breakthrough was and algorithm for maximum aposteriori estimate in binary Ising model using minimal cut in a suitable graph [2].

Similar methods can be applied also to other tasks: the filtration of grayscale image, segmentation and correspondence between two images for image registration, stereovision or optical flow estimation. Aim of this paper is demonstration of simple optimization method using graph cut for solution of fitration and registration tasks. The formulas are presented for square images with horizontal and vertical connectivity between pixels only, but they can be easilly generalized to arbitrary image shape, dimension or connectivity.

2 Solving $TV-\mathbf{L}^p$ regularization by steepest descent method

Let g be given digital image and u the regularized (smoothed) image

$$g : Pix \to \{0 \dots 255\}\,,\; u : Pix \to \{0 \dots 255\}\,,$$

where Pix is a lattice of image elements (pixels)

$$Pix = \{\{i,j\}\,; i,j = 1 \dots N\}\,.$$

Let Edg be the set of vertical or horizontal edges between neighbouring image elements

$$Edg = \{\{\{i,j\}\,,\{i+1,j\}\}\,; i = 1 \dots N-1, j = 1 \dots N\}$$

$$\cup \{\{\{i,j\},\{i,j+1\}\};i=1\dots N,j=1\dots N-1\}.$$

The cost function is

$$F_p(u)=\sum_{e\in Edg}|u_{e_1}-u_{e_2}|+\frac{1}{p\lambda}\sum_{x\in Pix}|u_x-g_x|^p$$

and u is argument of minimum of F_p.

The first term is discrete total variation $TV(u)$ and it penalizes rough images, the second term is a multiple of $p-$th power of $\mathbf{L}^p$ norm of difference of images, the parameter λ controls the ratio of the image smoothness and its similarity to the original image.

F_p is convex if $p\geq 1$ and strictly convex for $p>1$ and $F_p(u)\to\infty$ for $u\to\infty$. There exist a minimum and it is unique if $p>1$.

2.1 Example:

Let $\gamma>0$, S be set of area A, I_S is its indicator function, $g(x)=\gamma I_S(x)$, then $TV(g)=\gamma TV(I_S)$, $\|g\|_p^p=\gamma^p A$. Argument of minimum is 0 for $\frac{\gamma^p}{p\lambda}A=\frac{1}{p\lambda}\|g\|_p^p<TV(g)=\gamma TV(I_S)$ and g if oposit inequalty holds. The enequalty is euivalent to $\gamma^{p-1}A<p\lambda TV(I_S)$. $TV(I_S)$ is approximately proportioal to the perimeter of the object S. For $p=1$ the solution does not depend on γ, so the $TV-\mathbf{L}^1$ regularization removes the objects with big ratio of the perimeter to the area (eg. small objects) irrespectively to the objects' contrast.

2.2 Minimization of F_p by the steepest gradient descend

Let $\delta \in \mathbb{Z}$, I be the indicator function.

We choose $S \subseteq Pix$ in each step of the method so that $F_p\left(u + \delta I_S\right)$ is minimal and we replace u by $u + \delta I_S$ [1].

F is convex and so the iterations with $\delta = \pm 2^n$, for $n = 7, 6, \ldots 0$ as long as the function value decreases shell yield globalminimum in finite number of steps.

We find the set $S \subseteq Pix$ using minimal cut in oriented graph with set of vertices $Pix \cup \{source, sink\}$. Minimal value of $F_p\left(u + \delta I_S\right)$ is equal to the value of minimal cut between *source* and *sink*, where the values of edges are

$$x \in Pix:\ h\left(source, x\right) = \frac{1}{p\lambda}\left|u_x + \delta - g_x\right|^p,\ h\left(x, sink\right) = 0,$$

$$\{x, y\} \in Edg:\ h\left(x, y\right) = \left|u_x - u_y - \delta\right|.$$

The minimal graph cut can be found by the duality theorem (minimal cut between *source* and *sink* is equal to the maximal flow between *source* and *sink*) and procedures for maximal flow [3].

2.3 Nanoparticles detection

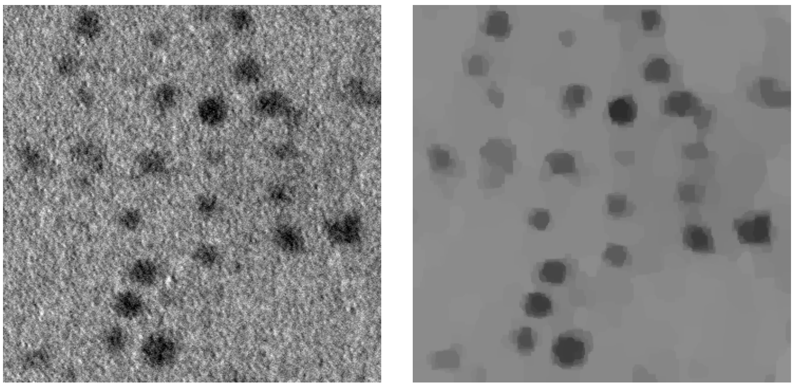


Image of silver nanoparticles (Silver proteinate, Fluka, Switzerland) before $TV - \mathbf{L}^1$ regularization and after it. Transmission electron microscopy images were provided by Anatolij Filimoněnko from IMG ASCR. Even the less contrast objects were preserved by the filtration and the resulting image is suitable for further processing and detection of particles by adaptive thresholding.

3 $TV - \mathbf{L}^1$ correspondence by steepest gradient descent

Let the u and v be the digital images between which we search he correspondence and let $\xi = \left(\xi^1, \xi^2\right)$ be the vector image of the correspondence relating element x of the image u and the element $x + \xi_x$ of the image v

$$u : Pix \to \mathbb{R},\ v : Pix \to \mathbb{R},$$

$$\xi : Pix \to \mathbb{Z}^2.$$

We use the function

$$F\left(\xi\right) = \sum_{e \in Edg} \left|\xi^1_{e_1} - \xi^1_{e_2}\right| + \left|\xi^2_{e_1} - \xi^2_{e_2}\right| + \frac{1}{\lambda} \sum_{x \in Pix} \left|u_x - v_{x+\xi_x}\right|$$

as the cost function of the correpondence image and ξ we choose as argument of the minimum of the function F.

3.1 Minimization of F by the steepest gradient descend

Let $\delta \in \mathbb{Z}^2$, $\delta = \left(\delta^1, \delta^2\right)$, I be the indicator function.

We choose set $S \subseteq Pix$ in each iteration step so that the value $F\left(\xi + \delta I_S\right)$ is minimal and replace ξ by $\xi + \delta I_S$. As the function F need not be convex, the iterations need not coverge to global minimum. δ we choose so, that its direction changes and its size decreases in subsequent iterations.

We find the $S \subseteq Pix$ using minimal cut in oriented graph with vertices $Pix \cup \{source, sink\}$ with values of edges:

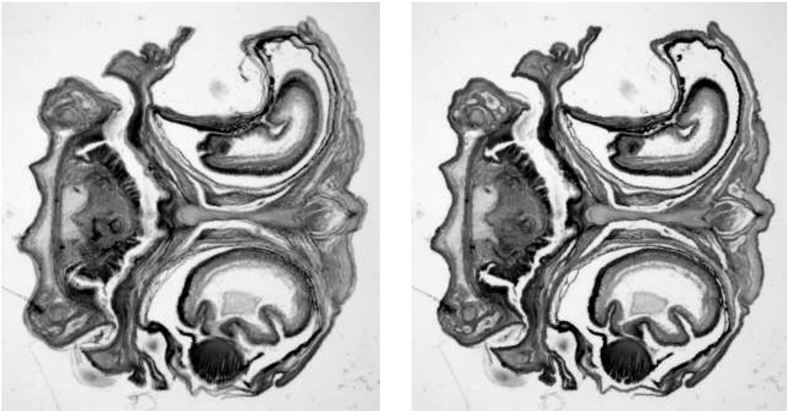
$$x \in Pix :\ h\left(source, x\right) = \frac{1}{\lambda} \left|u_x - v_{x+\xi_x+\delta}\right|,\ h\left(x, sink\right) = 0,$$

$$\{x, y\} \in Edg :\ h\left(x, y\right) = \left|\xi^1_x - \xi^1_y - \delta^1\right| + \left|\xi^2_x - \xi^2_y - \delta^2\right|.$$

The minimal cut can be found using the maximal flow between *source* and *sink* [3].

3.2 Registration of images of subsequent sections

For 3D reconstruction of an object from parallel physical sections the mutual position of the sections shall be found and eventual deformations caused by slicing shall be compensated.



Average of two images of physical sections of turtle head before and after registration. Images were provided by Barbara Tvarožková from faculty of science, Charles University. We can see sucessfull registration of the whole sections including discontinuous transition between the skull (right) and the jaw (left).

4 Conclusions

Combinatorial optimization based on minimal graph cut found recently lot of sucessfull applications in image analysis. The new method fixes a value of change and searches optimal set of changed pixels (that decreases most the cost function value) while the naive optimization searches optimal value of step for each pixel separately. The new method poses higher computational demands but is more resistant to local minima.

Reference

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