

## Optimal decoding and information transmission in Hodgkin-Huxley neurons under metabolic cost constraints

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Information theory quantifies the ultimate limits on reliable information transfer by means of the channel capacity. However, the channel capacity is known to be an asymptotic quantity, assuming unlimited metabolic cost and computational power. We investigate a single-compartment Hodgkin-Huxley type neuronal model under the spike-rate coding scheme and address how the metabolic cost and the decoding complexity affects the optimal information transmission. We find that the sub-threshold stimulation regime, although attaining the smallest capacity, allows for the most efficient balance between the information transmission and the metabolic cost. Furthermore, we determine post-synaptic firing rate histograms that are optimal from the information-theoretic point of view, which enables the comparison of our results with experimental data.

Keywords: Neuronal coding; Information transfer; Optimal decoding

### 1. INTRODUCTION

Neuroscience, in particular the neural coding problem, computer science and Shannon's information theory (Shannon and Weaver, 1949) are historically very closely related (Wiener, 1948; McCulloch and Pitts, 1943). In fact, the first studies on the information transfer in neurons appeared relatively shortly after Shannon published his theory (Quastler, 1953; MacKay and McCulloch, 1952). Early applications of the information theory to neurosciences, however, failed to provide the expected insight, mostly because of lack of understanding of the involved biological processes. Recently, as biologically relevant biophysical models of neurons are available and as experimental data from different sensory systems are gathered routinely, there is a marked revival of interest in information-theoretic methods (Milenkovic *et al.*, 2010; Dimitrov, Lazar, and Victor, 2011).

Information processing in the nervous system depends on the transformation of the graded membrane potential into a train of action potentials (Perkel and Bullock, 1968; Dayan and Abbott, 2001). The main characteristics of this transformation were established by extensive experimental investigations in sensory and central neurons and neuronal models describe them adequately (Hodgkin and Huxley, 1952; Gerstner and Kistler, 2002; Tuckwell, 1988). The neurons initially convert the external stimulus into an internal signal, i.e., the membrane depolarization. This signal, a continuous one, is then converted into the response, i.e., into the train of all-or-none pulses (the spike train). A satisfying description of how the information is represented in such spike trains is still not known (Perkel and Bullock, 1968; Stein, Gossen, and Jones, 2005). The classical approach relies on the rate coding

scheme (Adrian, 1928), where the information is contained in the number of action potentials in a selected time window (Fig. 1). Any information possibly encoded in the temporal structure of the spike train (Theunissen and Miller, 1995; Kostal, Lansky, and Rospars, 2007) is therefore ignored.

The neuronal response often varies, apparently randomly, across identical stimulus trials (Stein, Gossen, and Jones, 2005). The stochastic nature of the response makes the problem of reliable information transmission highly non-trivial and attractive from the information-theoretic point of view. The classical application of information theory considers the neuron (or a population of neurons) to act as an information channel (Stein, 1967; de Ruyter van Steveninck and Laughlin, 1996; Ikeda and Manton, 2009; Johnson, 2010; Chacron, Lindner, and Longtin, 2007). In many cases, motivated especially by the efficient coding hypothesis (Barlow, 1961), the goal is to provide the ultimate limits on neuronal performance in the point-to-point communication situations (Rieke *et al.*, 1997; Atick, 1992; Laughlin, 1981; de Ruyter van Steveninck and Laughlin, 1996; Kostal, Lansky, and Rospars, 2008). The communication process is described by means of mutual information between the neuronal inputs and responses, with channel capacity providing the upper bound on information transfer. However, in practical terms, there is a penalty to be paid in terms of delay and complexity of the information coding and decoding procedures (Gallager, 1968). Hence, the crucial question of information transfer *efficiency* arises.

In this paper we argue that the maximal (unconstrained) amount of information might not be the main objective: the energetic (metabolic) expenses and information encoding-decoding complexity (speed) should definitely be considered. After all, it is vital for neural systems to balance the “perception quality”, real-time performance and metabolic workload associated with the information processing. We believe that our attempt is timely, since the problem of metabolic ef-

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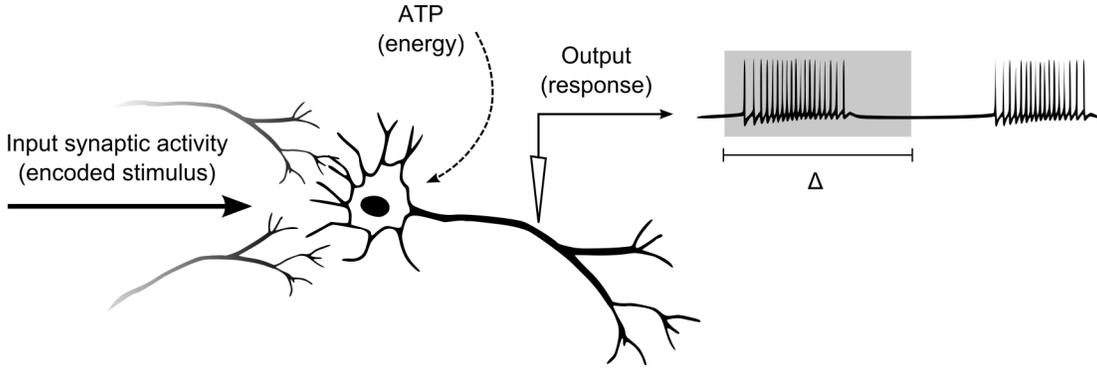


Figure 1. Situation investigated in this paper. The input to the model neuron is the excitatory and inhibitory firing activity of presynaptic neurons, representing the encoded stimulus feature. The output is the number of action potentials observed in a certain time window of length  $\Delta$  (we assume the frequency code). The question “*How much information* can at best be reliably transferred by such a system?” is answered by the concept of channel capacity within the Shannon’s information theory. Neurons, however, consume energy in order to maintain the resting membrane potential and to generate action potentials. Generally, it holds that higher firing rates tend to consume more energy than lower firing rates. Furthermore, information theory implies, that as the information flow approaches the ultimate Shannon limit, the required coding and decoding operations are likely to be algorithmically very complex and difficult to implement practically. In this paper we investigate the information-processing performance in a realistic Hodgkin-Huxley-type neuronal model with balanced input. We consider the infomax principle together with the optimal balance between the information transfer, the incurred metabolic cost and the algorithmic decoding complexity.

efficiency has been attracting attention in the theoretical neuroscience community recently (Suksompong and Berger, 2010; Sengupta, Laughlin, and Niven, 2013; Kostal, Lansky, and McDonnell, 2013; Kostal and Lansky, 2013), see also Sengupta and Stemmler (2014) and references therein. Additionally, the practically important problem of coding complexity (Zhong, Alajaji, and Campbell, 2006; Huang, Meyn, and Medard, 2006; Polyanskiy, Poor, and Verdu, 2010) has not been, to the best of our knowledge, addressed in the computational neuroscience literature so far.

## 2. METHODS

### 2.1. Neuronal model and metabolic cost

We consider a single-compartment neuron model driven by a synaptic current  $I_{\text{syn}}$  (Hodgkin and Huxley, 1952; Benda and Herz, 2003),

$$C_m \frac{dV}{dt} = -g_L(V - E_L) - I_{\text{Na}} - I_{\text{Kd}} + I_{\text{syn}}, \quad (1)$$

where  $C_m = 1 \mu\text{F}/\text{cm}^2$  is the membrane capacitance,  $V$  is the membrane depolarization,  $g_L = 0.1 \text{ mS}/\text{cm}^2$  is the leak conductance,  $E_L = -67 \text{ mV}$  is the reversal potential of the leak current, and  $I_{\text{Na}}$ ,  $I_{\text{Kd}}$  are the sodium, delayed rectifier potassium current, respectively.

The sodium current  $I_{\text{Na}}$  is given by

$$\begin{aligned} I_{\text{Na}} &= g_{\text{Na}} m^3 h (V - E_{\text{Na}}), \\ \frac{dm}{dt} &= \alpha_m(V)(1 - m) - \beta_m(V)m, \\ \alpha_m(V) &= \frac{-0.32(V + 54)}{\exp[-(V + 54)/4] - 1}, \\ \beta_m(V) &= \frac{0.28(V + 27)}{\exp[(V + 27)/5] - 1}, \\ \frac{dh}{dt} &= \alpha_h(V)(1 - h) - \beta_h(V)h, \\ \alpha_h(V) &= 0.128e^{-(V+50)/18}, \\ \beta_h(V) &= \frac{4}{1 + \exp[-(V + 27)/5]}, \end{aligned} \quad (2)$$

where  $g_{\text{Na}} = 100 \text{ mS}/\text{cm}^2$ ,  $E_{\text{Na}} = 50 \text{ mV}$ .

The delayed rectifier potassium current  $I_{\text{Kd}}$  is given by

$$\begin{aligned} I_{\text{Kd}} &= g_{\text{Kd}} n^4 (V - E_{\text{K}}), \\ \frac{dn}{dt} &= \alpha_n(V)(1 - n) - \beta_n(V)n, \\ \alpha_n(V) &= \frac{-0.032(V + 52)}{\exp[-(V + 52)/5] - 1}, \\ \beta_n(V) &= 0.5e^{-(V+57)/40}, \end{aligned} \quad (3)$$

where  $g_{\text{Kd}} = 80 \text{ mS}/\text{cm}^2$  and  $E_{\text{K}} = -100 \text{ mV}$ .

The point-conductance model proposed by Destexhe *et al.* (2001) was used to simulate realistic synaptic input. The synaptic input  $I_{\text{syn}}$  is given by

$$I_{\text{syn}}(t) = g_e(t)(E_e - V) + g_i(t)(E_i - V), \quad (4)$$

where  $g_{e(i)}$  and  $E_{e(i)}$  are the excitatory (inhibitory) synaptic conductances and reversal potentials, respectively. The synaptic conductances are described by the Ornstein-Uhlenbeck process (Tuckwell, 1988; Kobayashi, Shinomoto, and Lansky, 2011),

$$\frac{dg_e(t)}{dt} = -\frac{g_e(t) - \mu_e}{\tau_e} + \sigma_e \eta_e(t), \quad (5)$$

$$\frac{dg_i(t)}{dt} = -\frac{g_i(t) - \mu_i}{\tau_i} + \sigma_i \eta_i(t), \quad (6)$$

where  $\tau_{e(i)}$  is the time constant of the excitatory (inhibitory) synaptic conductance, and  $\eta_{e(i)}(t)$  are mutually independent Gaussian white noises with zero means and unit variances. The asymptotic mean and variance of the synaptic conductances are  $\mu_{e(i)}$  and  $\sigma_{e(i)}^2 \tau_{e(i)}/2$ , respectively. The standard deviations of the synaptic conductances are assumed to be proportional to the corresponding mean (Miura *et al.*, 2007), i.e.,  $\sigma_e = \mu_e/2$  and  $\sigma_i = \mu_i/4$ . The parameters are  $E_e = 0$  mV,  $E_i = -70$  mV,  $\tau_e = 2.7$  ms, and  $\tau_i = 10.5$  ms.

The *stimulus intensity*,  $x$ , is given by the mean excitatory synaptic conductance  $\mu_e$  during the time window of length  $\Delta$ . The mean inhibitory conductance is varied proportionally,  $\mu_i \propto \mu_e$ , in accordance with the balanced excitation and inhibition (Shadlen and Newsome, 1998; Miura *et al.*, 2007; Sengupta, Laughlin, and Niven, 2013). We define the effective reversal potential  $V_r$ , at which the mean synaptic current equals to zero,

$$\langle I_{\text{syn}} \rangle = \mu_e(V_r - E_e) + \mu_i(V_r - E_i) = 0. \quad (7)$$

The neuron model was simulated by using Euler-Maruyama integration scheme with a time step of 0.025 ms. Because the time step is much shorter than all intrinsic time constants of the model, a first-order method is sufficient. The time window  $\Delta$  determines the stimulus time scale. For the purpose of this paper we set  $\Delta = 0.5$  s so that the effect of past stimulation (before  $\Delta$ ) can be neglected.

The *response* of the neuron,  $y$ , is the number of spikes observed in  $\Delta$ , the corresponding firing rate is given as  $y/\Delta$ . It should be noted that the response  $y$  is stochastic and described by the random variable  $Y$ , because the synaptic conductances are described by the stochastic differential equation (Eqs. 5, 6).

The *metabolic cost*  $w$  of neuronal activity is the sum of *i*) cost to maintain resting potential and *ii*) cost of generating spikes. We follow Attwell and Laughlin (2001) and define the metabolic cost as the number of ATP molecules consumed in  $\Delta$ ,

$$w = \beta\Delta + \kappa y, \quad (8)$$

where  $\kappa = 0.71 \cdot 10^9$  ATP molecules is the cost of a single spike and  $\beta = 0.342 \cdot 10^9$  ATP molecules per second to maintain the neuronal resting potential. The model in Eq. (8) therefore assumes that the cost of each spike is constant, independent of the evoked firing rate. The approximate linearity of the metabolic cost  $w$  in the response firing rate  $y$  is supported by both experimental (Attwell and Laughlin, 2001) and theoretical studies (Balasubramanian and Berry, 2002). However, the

linearity is in no way essential for the methodology employed in this paper, and more detailed cost models (once available) can be readily implemented.

## 2.2. Information transmission and decoding

In order to analyze the information-transmission capabilities of the neuronal model described in the previous section, the *conditional* probability density function  $f(y|x)$  (Gallager, 1968) of the response  $Y = y$  given some particular input  $x$  must be known. It is impossible to obtain a closed-form solution for  $f(y|x)$  based on the Eqs. (1)–(7), therefore extensive computer simulations were used to obtain the numerical approximation to  $f(y|x)$ .

Assume that the individual stimulus intensities occur with different relative frequencies, i.e., the stimulus ensemble is described by the (continuous) random variable  $X$  with probability density function  $p(x)$ . The average metabolic cost induced by such  $X \sim p(x)$  is given as

$$W_p = \int w(x)p(x) dx, \quad (9)$$

where  $w(x)$  is the expected metabolic cost associated with the stimulus value  $X = x$ . Due to Eq. (8) we have

$$w(x) = \int (\beta\Delta + \kappa y)f(y|x) dy. \quad (10)$$

We consider the neuron to act as the *information channel*, in accordance with theoretical neuroscience literature (Johnson, 2010; Suksompong and Berger, 2010; Ikeda and Manton, 2009; Stein, 1967; de Ruyter van Steveninck and Laughlin, 1996; Kostal, Lansky, and McDonnell, 2013). That is, we assume that the task of the neuron is to transmit information about external stimulus to its post-synaptic neurons. The general methodology for determining the *information capacity* and the *capacity-cost* function in neuronal models, including the description of numerical algorithms, has been recently covered in Kostal, Lansky, and McDonnell (2013); Kostal and Lansky (2013). Hence, we briefly repeat the key equations for these two quantities without going into details.

The capacity  $C$  gives the maximal rate in bits at which information can be communicated reliably, with arbitrarily small error of encoding, through an unreliable channel,

$$C = \max_{p(x)} I(X; Y), \quad (11)$$

where

$$I(X; Y) = \iint p(x)f(y|x) \log_2 \frac{f(y|x)}{\int f(y|\tilde{x})p(\tilde{x}) d\tilde{x}} dy dx, \quad (12)$$

is the mutual information between stimuli and responses. The *capacity-cost* function,  $C(W)$ , gives the (constrained) maximum mutual information such that the metabolic cost  $W_p$  given by Eq. (9) does not exceed some selected value  $W$ , i.e.,

$$C(W) = \max_{p(x): W_p \leq W} I(X; Y). \quad (13)$$

Finally, we define the information-metabolic *efficiency*  $E$ , also known as the *capacity per unit cost* (Verdu, 1990; Kostal, Lansky, and McDonnell, 2013), that is the optimal balance between information and cost

$$E = \max_W \frac{C(W)}{W}. \quad (14)$$

Equivalently, one can describe  $1/E$  as the *minimum cost* of one bit of information achievable by the system. We define the *optimal cost*  $W^*$  as the value which achieves the maximum in Eq. (14),

$$W^* = \arg \max_W \frac{C(W)}{W}. \quad (15)$$

Besides the information-cost balance described above, another key factor for practical feasibility of information transfer is the algorithmic decoder *complexity* (Gallager, 1968; Polyanskiy, Poor, and Verdu, 2010), e.g., how computationally “hard” it is to read-out the information. The decoding procedure aims to estimate the correct input signal from the observed response. The capacity and capacity-cost function assume that infinite computing power is available at the decoder (McEliece, 2002), and from this point of view both these quantities are *asymptotic*. The methodology to explicitly involve the decoding complexity, and hence to consider the non-asymptotic regime of information transmission, is provided by the Gallager’s *maximum likelihood* (ML) decoding bound (Gallager, 1968), which we summarize below.

In order to transmit information as reliably as possible through unreliable channel, the procedure known as *channel coding* is essential. The idea is to process a sequence (vector) of stimuli values,  $\mathbf{x} = \{x_1, x_2, \dots, x_N\}$ , rather than just a single value, in order to reduce the effect of noise on the information transmission. Assume that  $M$  such distinct sequences,  $\{\mathbf{x}^{(i)}\}_{i=1}^M$ , each of length  $N$ , are used as input signals. The *rate*  $R$  of information transmission in bits per second is then defined as (Gallager, 1968)

$$R = \frac{\log_2 M}{N\Delta}. \quad (16)$$

The Shannon’s channel coding theorem guarantees that for all rates below the channel capacity,  $R < C$ , the probability of *decoding error* (i.e., the probability that the decoder fails to identify the correct input sequence) can be made arbitrarily small. The ML decoder chooses  $\mathbf{x}^{(\text{dec})}$  such that

$$f(\mathbf{y}|\mathbf{x}^{(\text{dec})}) \geq f(\mathbf{y}|\mathbf{x}^{(i)}), \quad i = 1, \dots, M \quad (17)$$

where  $\mathbf{y} = \{y_1, \dots, y_N\}$  is the observed response vector and  $f(\mathbf{y}|\mathbf{x}) = \prod_{i=1}^N f(y_i|x_i)$ . Note that the ML decoding corresponds to the minimum-error decoding if the sequences are equiprobable and hence Eq. (17) is often denoted as the optimal decoding rule (Gallager, 1968). The decoding error occurs when  $\mathbf{x}^{(\text{dec})} \neq \mathbf{x}^{(\text{in})}$ , where  $\mathbf{x}^{(\text{in})}$  is the true input signal. The Gallager’s bound guarantees that for some selected information rate  $R < C$  and a sequence length  $N$ , there exists a set of  $M$  inputs such that the average probability of decoding error does not exceed  $P_e$ , defined as

$$P_e = \exp[-NE_r(R)]. \quad (18)$$

The function  $E_r(R)$  is known as the *error exponent*, which is positive for  $R < C$ , and is given by

$$E_r(R) = \max_{0 \leq \varrho \leq 1} [\max_{p(x)} E_0(\varrho, p) - \varrho R], \quad (19)$$

$$E_0(\varrho, p) = -\ln \int \left[ \int f(y|x)^{1/(1+\varrho)} p(x) dx \right]^{1+\varrho} dy. \quad (20)$$

The maximization in Eq. (19) can be also constrained by the selected metabolic cost  $W$ , so that only  $p(x)$  such that  $W_p \leq W$  are considered. Such a constraint does not restrict individual  $N$ -sequences though, which would require a more complicated optimization beyond the scope of this effort (Gallager, 1968). In this paper we fix the value of  $P_e$  and look for the desired sequence length  $N^*$ , obtained by manipulating Eq. (18) as

$$N^* = \left\lceil -\frac{\ln P_e}{E_r(R)} \right\rceil. \quad (21)$$

The decoding complexity of the ML decoder grows exponentially with  $N$  (Caire, Shamai, and Verdu, 2004; Gallager, 1968). We therefore define, in analogy with the information-cost *efficiency* in Eq. (14), the *information-cost-complexity efficiency* as

$$\Gamma = \max_{R < C(W)} \frac{R}{W \exp N^*}. \quad (22)$$

In other words, the goal is to maximize the information rate  $R$ , minimize the cost  $W$  and minimize the decoding complexity  $\exp N^*$ .

The solution to Eq. (19) is usually not available in a closed form, similarly to the mutual information maximization in Eq. (11). However, since the determination of  $E_r(R)$  is convex in  $p(x)$ , we use the cutting-plane linear programming algorithm described in Huang, Meyn, and Medard (2006); Kostal and Lansky (2013).

### 3. RESULTS

Three stimulation regimes are distinguished based on the value of  $V_r$  in Eq. (7):

- the *sub-threshold* regime,  $V_r = -70$  mV,
- the *critical* regime,  $V_r = -60$  mV,
- the *supra-threshold* regime,  $V_r = -50$  mV.

The stimulus intensity,  $x = \mu_e$ , in these three cases takes values in the range from 0.01 to 0.3 mS/cm<sup>2</sup>.

Fig. 2 visualizes the stimulus-response behavior of the investigated neuronal model together with the summary of our main results. Fig. 2a shows the rate-intensity function,  $\mu(x) = \int y f(y|x) dy$ , in the sub-threshold case together with the standard deviation of the response firing rate across trials. The color indicates the probability  $f(y|x)$  that a particular firing rate is observed for given stimulus intensity.

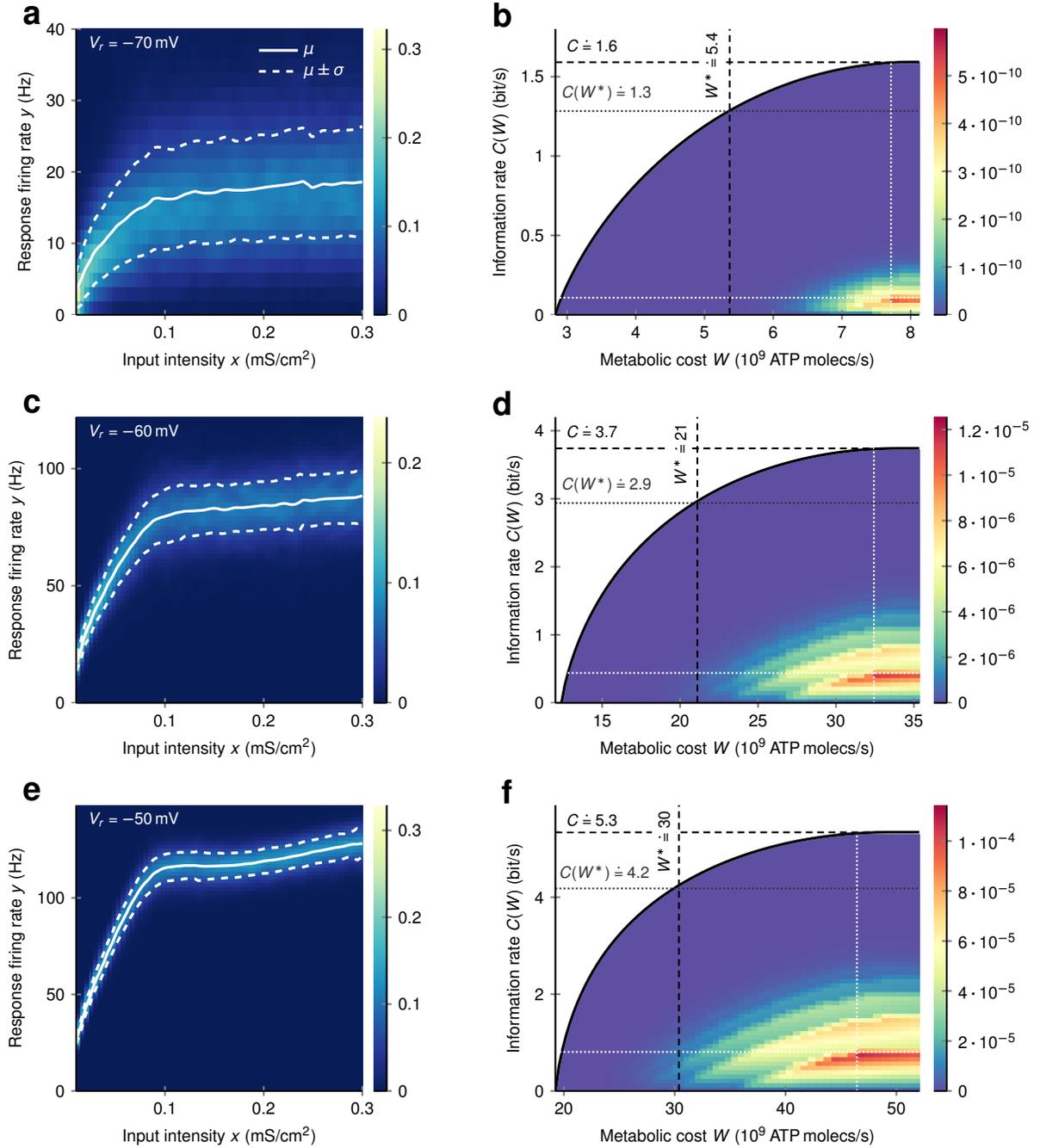


Figure 2. Information-optimality conditions in the Hodgkin-Huxley neuronal model with point-conductance balanced input. **(a)** Stimulus-response relationship of the investigated neuronal model in the sub-threshold stimulation regime. The color indicates the probability density of observing particular firing rate given the stimulus intensity (excitatory conductance). The rate-intensity function (solid white line) accompanied by the response standard deviation (dashed) are indicated for convenience. **(b)** The capacity-cost function  $C(W)$  (solid) determines the maximum information in bits per second that can be transferred by the neuron if the metabolic cost does not exceed  $W$  ATP molecules per second. The maximum of  $C(W)$ , disregarding the cost, is the traditional channel capacity  $C$  (horizontal dashed). The maximal information transfer per metabolic cost occurs at  $W^*$  (vertical dashed), see Fig. 3. The color indicates the value of the ratio  $R/(W \exp N^*)$  in Eq. (22) for the average probability of decoding error  $P_e = 10^{-10}$ , describing the balance between information, cost and algorithmic complexity under the optimal maximum likelihood decoding rule. Higher values mean better performance, the maximum ( $\Gamma$ ) is indicated by white dotted lines. **(c, d)** Results for the critical stimulation regime. **(e, f)** Results for the supra-threshold stimulation regime.

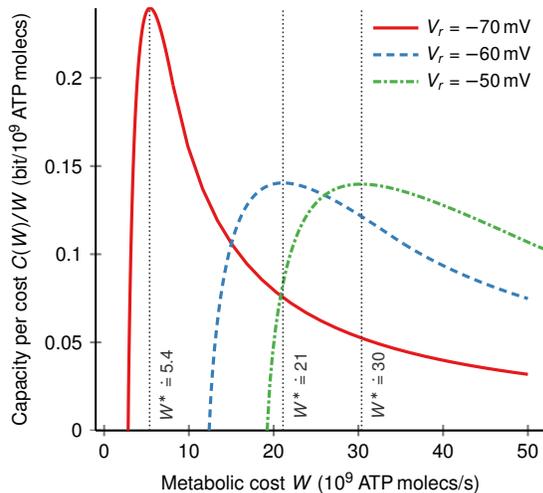


Figure 3. Information per metabolic cost ratio,  $C(W)/W$ . All three investigated cases (sub-, critical and supra-threshold) are shown. It follows that the sub-threshold regime (solid) greatly outperforms (in bits per ATP consumption) the other two: the value of  $E$  (maximum of  $C(W)/W$ ) is larger and the required metabolic cost smaller. The efficiency of the critical (dashed) and supra-threshold (dashed-dotted) regimes is comparable, however, the required metabolic cost is larger in the supra-threshold case.

The capacity-cost function  $C(W)$  is shown in Fig. 2b (solid line). The channel capacity is the maximum of  $C(W)$  as the metabolic cost is allowed to grow without bounds. In this case  $C \doteq 1.6$  bit/s. The optimal metabolic cost, in Eq. (15) is  $W^* = 5.4 \times 10^9$  ATP molecules per second, which is indicated by the vertical dashed line. The efficiency itself equals  $E \doteq 0.24$  bit per  $10^9$  ATP molecules. The color indicates the value of the information-cost-complexity ratio,  $R/(W \exp N^*)$ , for all rates below  $C(W)$  and the average probability of decoding error not exceeding  $P_e = 10^{-10}$ . The point of maximal efficiency defined by Eq. (22) is marked by white dotted lines and  $\Gamma \doteq 6 \times 10^{-10}$  bit per  $10^9$  ATP molecules per arithmetic operation. Similarly, the stimulus-response properties and information-optimality results for the critical case are shown in Fig. 2c, d and for the supra-threshold case in Fig. 2e, f. Overall, the values of the actual information rates in the metabolically efficient regime are about 80 % of the channel capacity; with the decoding complexity included the rates drop below 20 % of  $C$ . The values of  $C, E$  and  $\Gamma$  for each stimulation regime are summarized in Table 1.

In order to gain more insight into the information-metabolic cost trade-off we show the ratio  $C(W)/W$  in dependence on  $W$  for each stimulation regime in Fig. 3. Note that the maximum of  $C(W)/W$  corresponds to the efficient point  $E$  as defined in Eq. (14).

Based on Figs. 2, 3 and Table 1 we see, that the sub-threshold regime attains the smallest capacity value, while the supra-threshold has the largest. However, from the point of view of information-cost efficiency  $E$ , the sub-threshold regime is the most efficient one. The efficiency is approximately the same for the critical and supra-threshold regimes,

note though that the optimal cost is largest in the supra-threshold case. The metabolic efficiency is explained by the relatively low response firing rate of the sub-threshold neuron, which is nonetheless accompanied by a significant amount of response variability. In order to achieve reliable information transmission in such a case, more complex coding operations are required. In fact, the information-cost-complexity efficiency  $\Gamma$  shows that the coding complexity has an important effect on the optimality conditions. Similarly to the case of capacity, the performance (value of  $\Gamma$ ) decreases with decreasing  $V_r$ . However, unlike the capacity case, the gap between the sub-threshold and critical regimes is substantial, amounting to five orders of magnitude. Also note that the required sequence length  $N^*$  grows very rapidly as the rate  $R$  increases (Gallager, 1968). Consequentially, the ratio  $R/(W \exp N^*)$  decays to zero quickly with increasing rates, as seen in Fig. 2b, d, f.

The information-optimality results presented so far do not include quantities that can be directly observed in experiments. However, we can calculate the optimal post-synaptic firing rate distributions  $f^*(y)$ ,

$$f^*(y) = \int f(y|x)p^*(x) dx, \quad (23)$$

which can be easily compared against experimental data. The input distributions  $p^*(x)$  that maximize the investigated quantities  $C$ ,  $E$ , and  $\Gamma$  (Eqs. 11, 14 and 22) are typically discrete with finite points of support (Smith, 1971; Abou-Faycal, Trott, and Shamai, 2001; Ikeda and Manton, 2009; Kostal, Lansky, and McDonnell, 2013).

The respective shapes of  $f^*$  are shown in Fig. 4, together with the matching stimulus distributions  $p^*(x)$ . Each column represents the stimulation regime, sub-threshold in Fig. 4a, d, g, critical in Fig. 4b, e, h, supra-threshold in Fig. 4c, f, i. The first row (a-c) results from the capacity-achieving ( $C$ ) input distributions (the infomax principle), the second row (d-f) represent the information-cost efficiency ( $E$ ) achieving point. Finally, the third row (g-i) results from the optimal balance between information transmission, metabolic cost and coding complexity ( $\Gamma$ ). Overall, the  $C$ -distributions are spread across the whole range of firing rates with modes at small firing rates where the relative response variability tends to be small (Fig. 2). The  $E$ -distributions are right skewed and unimodal with a strong suppression of high firing rates as a consequence of the metabolic cost constraint. The  $\Gamma$ -distribution for the supra-threshold case (Fig. 4i) is highly multimodal, while the sub-threshold case (g) shows little difference with respect to the  $C$ -distribution (a). The reason for multimodality follows from the fact that the  $\Gamma$ -achieving  $p^*(x)$  tend to be more sparse, with less discrete stimuli intensities allowed. Such sparse  $p^*(x)$  generate channel codes which are potentially easier to decode (Gallager, 1968; Huang, Meyn, and Medard, 2006). It is important to note that such codes may optimally communicate information at much lower rates than would follow from the mutual information calculation for such  $X \sim p^*(x)$ , i.e., it typically holds  $R < I(X; Y)$ .

Table 1. Numerical values of the maximal (optimal) information quantities for the three stimulation regimes. The information capacity  $C$  (Eq. 11) gives the maximum rate for reliable information transfer. The information-cost efficiency  $E$  (Eq. 14) similarly describes the maximum rate per unit metabolic cost. The information-cost-complexity efficiency  $\Gamma$  (Eq. 22) extends these concepts further by including the algorithmic complexity of the decoding procedure into the balance.

Model	Capacity ( $C$ )	Information-cost eff. ( $E$ )	Information-cost-complexity eff. ( $\Gamma$ )
	[bit/s]	[bit/( $10^9$ ATP moles)]	[bit/( $10^9$ ATP moles)/operation]
Sub-threshold	1.59	0.24	$6 \times 10^{-10}$
Critical	3.71	0.14	$1.2 \times 10^{-5}$
Supra-threshold	5.29	0.14	$1.1 \times 10^{-4}$

#### 4. DISCUSSION

It is tempting from the efficient coding hypothesis point of view (Barlow, 1961) to identify the capacity with the optimal point of neuronal performance, e.g., the infomax hypothesis (Linsker, 1988), and the capacity-achieving input distribution with the matching stimulus statistics. Continuous input distributions, resulting either from the deterministic approach (Laughlin, 1981; Kostal, Lansky, and Rospars, 2008), from the classical Gaussian channel application (de Ruyter van Steveninck and Laughlin, 1996) or from the low-noise approximation (Nadal and Parga, 1994; McDonnell and Stocks, 2008), potentially resemble the natural stimuli distributions. On the other hand, interpreting the exactly optimal discrete input distributions (Ikeda and Manton, 2009; Kostal, Lansky, and McDonnell, 2013) as the natural stimulation statistics might lead to difficulties. The mutual information, capacity and the optimizing input distribution have precise interpretation under specific assumptions within the framework of Shannon’s theory (Gallager, 1968). We show, by adhering to the standard information-theoretic interpretation, that the capacity-achieving input distributions should not be confused with optimal stimulus statistics.

The information theory puts great emphasis on the process of *source* and *channel coding*, and on the probabilistic description of both the information source and the channel (Gallager, 1968). The notion of “coding” is employed in a different sense from the notion of coding in neuroscience community. The “neuronal coding” usually describes the way information is represented in the neuronal signal, e.g., should the estimator, depend on the firing rate or on the precise temporal sequence of spikes (Kostal, Lansky, and Rospars, 2007; Perkel and Bullock, 1968; Kobayashi *et al.*, 2013; Theunissen and Miller, 1995)? On the other hand, the *source coding* is concerned with the most efficient representation of the source in some convenient alphabet, typically binary in engineering applications, which in turn “measures” the information content of the source in bits. The representation can be either exact, unique for each possible stimulus intensity, or with a fidelity criterion which assigns the same representation to different stimuli (denoted as lossy compression in engineering). Typically, continuous-valued stimuli cannot be represented by a finite amount of bits and the lossy compression scheme must be used. The *channel coding* then prepares the source-encoded

information to be sent across the channel in such a way as to minimize the impact of channel noise. After the channel output is received, the channel input is estimated in the process known as *channel decoding*. The whole process is separated into two stages, *i*) the source and *ii*) the channel coding/decoding. This so called *separation assumption* is one of the key requirements for interpreting the notions of channel capacity and the optimal input distribution meaningfully, see, e.g. Gallager (1968, p.116) or Zhong, Alajaji, and Campbell (2006); Rimoldi (2002) for details. If we interpret the capacity calculated in this paper in its operational meaning, as the upper bound on reliable information transfer, we inevitably conclude, that the optimizing input density *does not* correspond to the optimal natural stimulation statistics, but to the optimal selection of the channel inputs employed for the channel coding procedure.

At this moment we did not consider the biological implementation of the coding procedures; whether and how the maximum likelihood decoder might be implemented by neurons and their networks (Deneve, Latham, and Pouget, 1999). We consider the input to our model neuron to be already processed (properly encoded) for the information transfer by the presynaptic neurons, while the response of the neuron is decoded by other neurons. Hence, we employ in this paper two classes of constraints. First, the metabolic cost constraint directly affects the studied neuron, since different firing rates consume different amounts of ATP molecules. Second, the decoding aspect (complexity) is delegated to other neurons, and possibly their networks, whose spiking activity is not part of our calculations and whose task is to read out the encoded message.

#### 5. CONCLUSIONS

By employing the standard approaches of information theory (Gallager, 1968) we calculated numerically the information-transmission capabilities of the Hodgkin-Huxley type of neuronal model. We determined both the ultimate asymptotic limits on information transmission (capacity and information-cost efficiency) and the non-asymptotic optimality conditions, by introducing the metabolic cost and the complexity of decoding operations.

We found that the sub-threshold stimulation regime offers

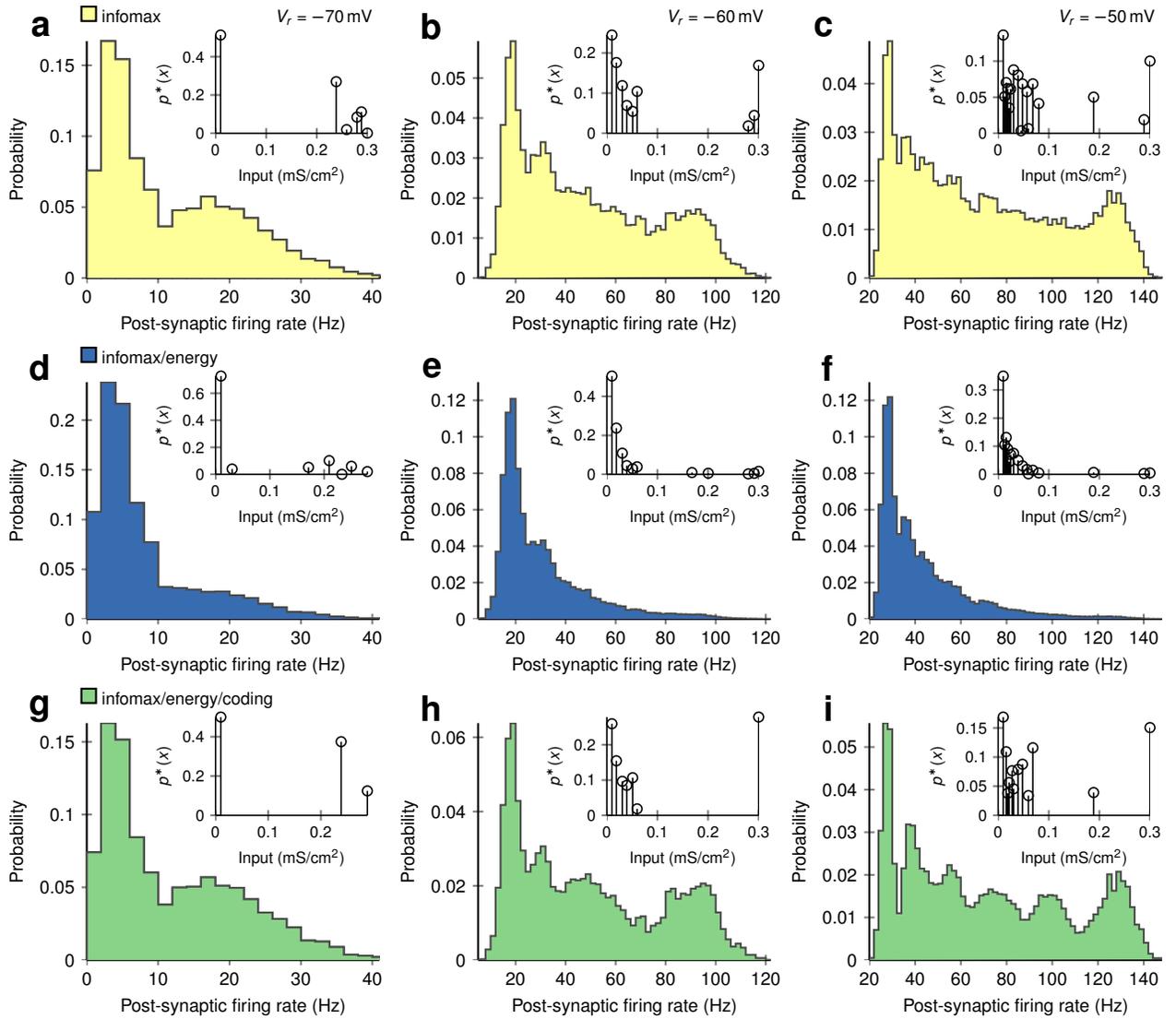


Figure 4. The post-synaptic firing rate histograms and the corresponding input distributions under different information-optimality conditions. (a, b, c) The capacity ( $C$ ) achieving firing rate distribution for the sub-threshold (a), critical (b) and supra-threshold (c) regimes. (d, e, f) The firing rate distributions under the optimal balance between the information transfer and the associated metabolic cost [achieving the efficiency point  $E$  in Eq. (14)]. In all three stimulation regimes [sub-threshold (d), critical (e) and supra-threshold (f)] the shape is unimodal and right skewed. (g, h, i) Distributions resulting from the optimal balance between information rate, metabolic cost and coding complexity [achieving the efficiency point  $\Gamma$  in Eq. (22)], for the corresponding stimulation regimes. Note in particular the expressed multimodality in the supra-threshold case (i), a consequence of the sparseness of the  $\Gamma$ -achieving input distribution.

the best asymptotic compromise between information transfer and energy consumption. The predicted post-synaptic firing rate distributions are sufficiently distinguishable under the investigated optimality conditions (infomax, energy-balance and decoding complexity balance), and offer the potential for future comparison with experimental data. Once the decoding cost is taken into account, the sparseness of the optimal input distributions result into highly multimodal response firing rate distributions.

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- Abou-Faycal, I. C., Trott, M. D., and Shamai, S., "The capacity of discrete-time memoryless Rayleigh-fading channels," *IEEE Trans. Inf. Theory* **47**, 1290–1301 (2001).
- Adrian, E. D., *Basis of Sensation* (W. W. Norton and Co., New York, 1928).
- Atick, J. J., "Could information theory provide an ecological theory of sensory processing?" *Netw. Comput. Neural Syst.* **3**, 213–251 (1992).
- Attwell, D. and Laughlin, S. B., "An energy budget for signaling in the grey matter of the brain," *J. Cereb. Blood Flow Metab.* **21**, 1133–1145 (2001).
- Balasubramanian, V. and Berry, M. J., "A test of metabolically efficient coding in the retina," *Netw. Comput. Neural Syst.* **13**, 531–552 (2002).
- Barlow, H. B., "Possible principles underlying the transformation of sensory messages," in *Sensory Communication*, edited by W. Rosenblith (MIT Press, Cambridge, 1961) pp. 217–234.
- Benda, J. and Herz, A. V. M., "A universal model for spike-frequency adaptation," *Neural Comput.* **15**, 2523–2564 (2003).
- Caire, G., Shamai, S., and Verdu, S., "Noiseless Data Compression with Low-Density Parity-Check Codes," in *Advances in Network Information Theory*, edited by P. Gupta, G. Kramer, and A. J. van Wijngaarden (DIMACS Series in Discrete Mathematics and Theoretical Computer Science, American Mathematical Society, 2004) pp. 263–284.
- Chacron, M. J., Lindner, B., and Longtin, A., "Threshold fatigue and information transfer," *J. Comput. Neurosci.* **23**, 301–311 (2007).
- Dayan, P. and Abbott, L. F., *Theoretical Neuroscience: Computational and Mathematical Modeling of Neural Systems* (MIT Press, 2001).
- Deneve, S., Latham, P. E., and Pouget, A., "Reading population codes: a neural implementation of ideal observers," *Nat. Neurosci.* **2**, 740–745 (1999).
- Destexhe, A., Rudolph, M., Fellous, J. M., and Sejnowski, T. J., "Fluctuating synaptic conductances recreate in vivo-like activity in neocortical neurons," *Neuroscience* **107**, 13–24 (2001).
- Dimitrov, A. G., Lazar, A. L., and Victor, J. D., "Information theory in neuroscience," *J. Comput. Neurosci.* **30**, 1–5 (2011).
- Gallager, R. G., *Information Theory and Reliable Communication* (John Wiley and Sons, Inc., New York, USA, 1968).
- Gerstner, W. and Kistler, W. M., *Spiking Neuron Models: Single Neurons, Populations, Plasticity* (Cambridge University Press, Cambridge, 2002).
- Hodgkin, A. L. and Huxley, A. F., "Propagation of electrical signals along giant nerve fibres," *Proc. Roy. Soc. B* **140**, 177–183 (1952).
- Huang, J., Meyn, S., and Medard, M., "Error exponents for channel coding with application to signal constellation design," *IEEE J. Select. Areas Commun.* **24**, 1647–1661 (2006).
- Ikeda, S. and Manton, J. H., "Capacity of a single spiking neuron channel," *Neural Comput.* **21**, 1714–1748 (2009).
- Johnson, D. H., "Information theory and neural information processing," *IEEE Trans. Inf. Theory* **56**, 653–666 (2010).
- Kobayashi, R., Namiki, S., Kanzaki, R., Kitano, K., Nishikawa, I., and Lansky, P., "Population coding is essential for rapid information processing in the moth antennal lobe," *Brain Res.* **1536**, 88–96 (2013).
- Kobayashi, R., Shinomoto, S., and Lansky, P., "Estimation of time-dependent input from neuronal membrane potential," *Neural Comput.* **23**, 3070–3093 (2011).
- Kostal, L. and Lansky, P., "Information capacity and its approximations under metabolic cost in a simple homogeneous population of neurons," *BioSystems* **112**, 265–275 (2013).
- Kostal, L., Lansky, P., and McDonnell, M. D., "Metabolic cost of neuronal information in an empirical stimulus-response model," *Biol. Cybern.* **107**, 355–365 (2013).
- Kostal, L., Lansky, P., and Rospars, J.-P., "Review: neuronal coding and spiking randomness," *Eur. J. Neurosci.* **26**, 2693–2701 (2007).
- Kostal, L., Lansky, P., and Rospars, J.-P., "Efficient olfactory coding in the pheromone receptor neuron of a moth," *PLoS Comput. Biol.* **4**, e1000053 (2008).
- Laughlin, S. B., "A simple coding procedure enhances a neuron's information capacity," *Z. Naturforsch.* **36**, 910–912 (1981).
- Linsker, R., "Self-organization in a perceptual network," *IEEE Comp.* **21**, 105–117 (1988).
- MacKay, D. M. and McCulloch, W. S., "The limiting information capacity of a neuronal link," *Bull. Math. Biol.* **14**, 127–135 (1952).
- McCulloch, W. S. and Pitts, W., "A logical calculus of the ideas immanent in nervous activity," *Bull. Math. Biophys.* **5**, 115–133 (1943).
- McDonnell, M. D. and Stocks, N. G., "Maximally informative stimuli and tuning curves for sigmoidal rate-coding neurons and populations," *Phys. Rev. Lett.* **101**, 058103 (2008).
- McEliece, R. J., *The Theory of Information and Coding* (Cambridge University Press, Cambridge, UK, 2002).
- Milenkovic, O., Alterovitz, G., Battail, G., Coleman, T. P., Hagenauer, J., Meyn, S. P., Price, N., Ramoni, M. F., Shmulevich, I., and Szpankowski, W., "Introduction to the special issue on information theory in molecular biology and neuroscience," *IEEE Trans. Inf. Theory* **56**, 649–652 (2010).
- Miura, K., Tsubo, Y., Okada, M., and Fukai, T., "Balanced excitatory and inhibitory inputs to cortical neurons decouple firing irregularity from rate modulations," *J. Neurosci.* **27**, 13802–13812 (2007).
- Nadal, J.-P. and Parga, N., "Nonlinear neurons in the low-noise limit: a factorial code maximizes information transfer," *Netw. Comput. Neural Syst.* **5**, 565–582 (1994).
- Perkel, D. H. and Bullock, T. H., "Neural coding," *Neurosci. Res. Prog. Sum.* **3**, 405–527 (1968).
- Polyanskiy, Y., Poor, V. H., and Verdu, S., "Channel coding rate in the finite blocklength regime," *IEEE Trans. Inf. Theory* **56**, 2307–2359 (2010).
- Quastler, H., *Essays on the Use of Information Theory in Biology* (University of Illinois Press, 1953).
- Rieke, F., de Ruyter van Steveninck, R., Warland, D., and Bialek, W., *Spikes: Exploring the Neural Code* (MIT Press, Cambridge, 1997).
- Rimoldi, B., "Beyond the separation principle: a broader approach to source-channel coding," in *4th Int. ITG Conf.* (VDE Verlag, Berlin, 2002) pp. 233–238.
- Sengupta, B., Laughlin, S. B., and Niven, J. E., "Balanced excitatory and inhibitory synaptic currents promote efficient coding and metabolic efficiency," *PLoS Comput. Biol.* **9**, e1003263 (2013).
- Sengupta, B. S. and Stemmler, M. B., "Power consumption during neuronal computation," *Proc. IEEE* **102**, 738–750 (2014).
- Shadlen, M. N. and Newsome, W. T., "The variable discharge of cortical neurons: Implications for connectivity, computation, and information coding," *J. Neurosci.* **18**, 3870–3896 (1998).
- Shannon, C. E. and Weaver, W., *The Mathematical Theory of Communication* (University of Illinois Press, Illinois, 1949).
- Smith, J. G., "The information capacity of amplitude-and variance-constrained scalar gaussian channels," *Inform. Control* **18**, 203–219 (1971).
- Stein, R. B., "The information capacity of nerve cells using a frequency code," *Biophys. J.* **7**, 797–826 (1967).
- Stein, R. B., Gossen, E. R., and Jones, K. E., "Neuronal variability: noise or part of the signal?" *Nat. Rev. Neurosci.* **6**, 389–397 (2005).
- de Ruyter van Steveninck, R. R. and Laughlin, S. B., "The rate of information transfer at graded-potential synapses," *Nature* **379**, 642–644 (1996).
- Suksompong, P. and Berger, T., "Capacity analysis for integrate-and-fire neurons with descending action potential thresholds," *IEEE Trans. Inf. Theory* **56**, 838–851 (2010).
- Theunissen, F. and Miller, J. P., "Temporal encoding in nervous systems: A rigorous definition," *J. Comput. Neurosci.* **2**, 149–162 (1995).
- Tuckwell, H. C., *Introduction to Theoretical Neurobiology, Vol. 2* (Cambridge University Press, New York, 1988).
- Verdu, S., "On channel capacity per unit cost," *IEEE Trans. Inf. Theory* **36**, 1019–1030 (1990).
- Wiener, N., *Cybernetics: Or Control and Communication in the Animal and the Machine* (John Wiley and Sons, New York, 1948).
- Zhong, Y., Alajaji, F., and Campbell, L. L., "On the joint source-channel coding error exponent for discrete memoryless systems," *IEEE Trans. Inf. Theory* **52**, 1450–1468 (2006).