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ESTIMATING LENGTH AND SURFACE AREA BY SYSTEMATIC PROJECTIONS

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ABSTRACT

The study is aimed to estimators of length in 2-D or length and surface area in 3-D by systematic projections of randomly oriented objects to subspaces of lower dimensions. Ellipses, ellipsoids and segments are used as model objects with various degrees of simple anisotropy. The variation coefficient and distributions of estimates of size of the model objects by projections to regular sets of directions in 2-D and 3-D are evaluated. Estimators optimal in the case of totally anisotropic objects are designed.

Keywords: Crofton formula, length, surface area, systematic projections.

INTRODUCTION

Design-based methods of estimation of length in the plane or length and surface area in the space use projections of measured objects and Crofton formulas. The precision of the methods can be improved by using weighted average of several systematically chosen estimators (Mattfeldt *et al.*, 1985) decreasing the variance of the estimate due to the negative covariance of projections directions (antithetic effect). The variation coefficient (CV, the standard deviation divided by the mean) of the estimates depends on the degree of anisotropy of the measured object; the highest is for totally anisotropic objects. Models of objects with simple anisotropy are ellipses and rotationally symmetric ellipsoids - spheroids.

EQUIDISTANT DIRECTIONS IN THE PLANE

Variation coefficient of an estimator of a segment length by the projections into n equidistant directions CV_n can be calculated after Moran (1966, $\beta = \pi/(2n)$):

$$CV_n^2 = \left(\beta/\tan\beta + (\beta/\sin\beta)^2\right)/2 - 1 . \quad (1)$$

The above result as well as the distribution density of the estimate divided by the true segment length:

$$f(x) = \left(\beta \cdot \sqrt{(\beta/\sin\beta)^2 - x^2}\right)^{-1}, \quad x \in (\beta/\tan\beta, \beta/\sin\beta) \quad (2)$$

follows from an explicit expression of the estimator value (Steinhaus, 1930, Appendix); f is monotone and has a pole at the maximum value of the argument.

The variation coefficient of estimates of size of partially anisotropic objects, such as ellipse (Fig. 1), increases from zero to values calculated after (1). The more are equidistant directions of projections, the faster the variation coefficient tends to 0 if the ratio of the lengths of ellipse axes tends to 1. An asymptotic formula follows from Taylor series of CV in powers of eccentricity:

$$CV_n \cong \frac{(2n-3)!!}{2n\,!!}\frac{\sqrt{2}}{2^n}\left(\ln \frac{a}{b}\right)^n ; \quad \frac{a}{b} \to 1 \quad \left(k!! \equiv k(k-2)\ldots\right) \quad (3)$$

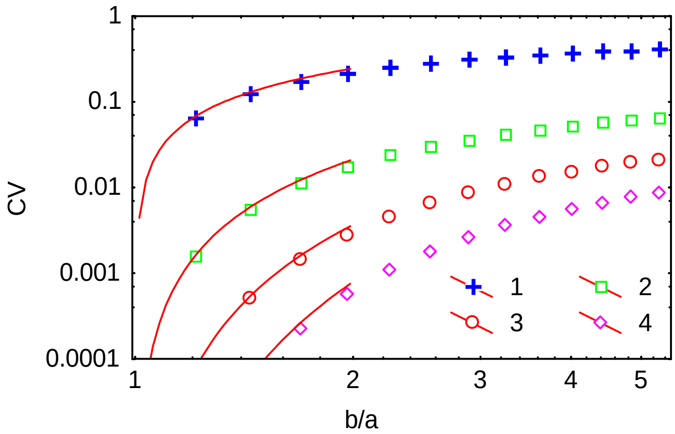


Fig. 1. *Variation coefficient of the estimate of the ellipse length by projections into 1 to 4 equidistant directions in plane; a, b are lengths of the ellipse axes. Full line: asymptotic approximation (3), limit at infinity - (1).*

SYSTEMATIC DIRECTIONS IN THE SPACE

Correlation coefficient of projections of a totally anisotropic object in the d-dimensional space into directions with angle ψ is:

$$K_d(\psi) = c_d^2 \, {}^{2}\!/_{d\pi} \left(\sin\psi + \left({}^{\pi}\!/_{2} - \psi\right)\cdot\cos\psi\right) - 1 \tag{4}$$

$$c_d = \frac{d\,\kappa_d}{2\kappa_{d-1}}, \quad \kappa_d = \frac{\pi^{d/2}}{\Gamma\left({}^{d}\!/_{2} - 1\right)}, \quad c_2 = \frac{\pi}{2},\ c_3 = 2$$

(Janáček, 1999). Expression (4) follows from the covariance of the Buffon needle projections in the plane (Schuster 1974) and the value of the mean square of unit vector projection to plane in R_d equal to 2/d. The estimator of length or surface area by projections into directions u_i, i = 1 to n is:

$$\text{est} = c_d \cdot \sum_{i=1}^{n} w_i P_{u_i}, \quad \sum_{i=1}^{n} w_i = 1 \tag{5}$$

and variation coefficient of the estimate of size of the totally anisotropic object is:

$$\text{CVest} = \sqrt{\sum_{i,j=1}^{n} w_i w_j K_d(\psi_{ij})}, \quad \cos(\psi_{ij}) = (u_i, u_j). \tag{6}$$

The only regular sets of directions in 3-D are a single direction, a pair of perpendicular directions and the normals to regular Platon solids: cube, octahedron, dodecahedron and icosahedron (n = 3, 4, 6, 10). Two series of sets of n = km systematic directions (rectangular and interleaved), are generated as follows:

$$u_{ij} = \left(\cos\omega_{ij}\cos\theta_i, \sin\omega_{ij}\cos\theta_i, \sin\theta_i\right), \ i = 1..k,\ j = 1..m \tag{7}$$

$$\sin\theta_i = (i - 0.5)/k, \text{ rectangular: } \omega_{ij} = 2\pi j/m \text{ or interleaved: } \omega_{ij} = (2j+i)\pi/m$$

The values of variation coefficient of sets with highest effectivity $1/(3nCV^2)$ from all sets from the series with the same or lower number of directions (n) are plotted in Fig. 2. The sets formed by vertices of triangulations of the sphere derived from icosahedron (Fuller polyhedra) and weighted by sum of areas of adjacent triangles yield still lower values of CV.

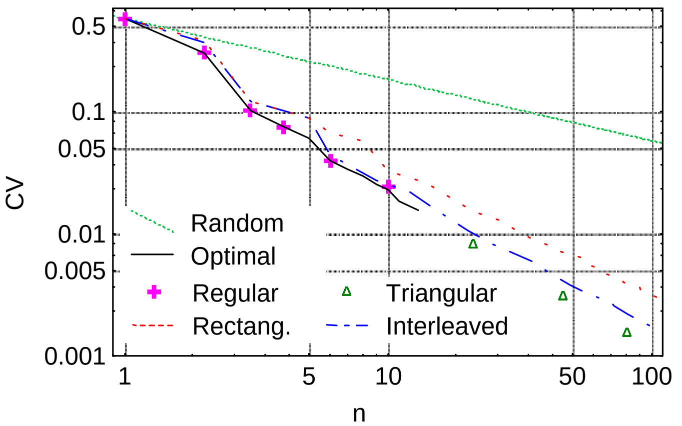


Fig. 2. *Variation coefficient of estimate by n systematic projections of totally anisotropic object in space. Random directions (average value), normals to regular polyhedra, Fuller triangulation of the sphere, best rectangular and interleaved systematic directions (7) and optimal sets for n = 3 to 13 are shown.*

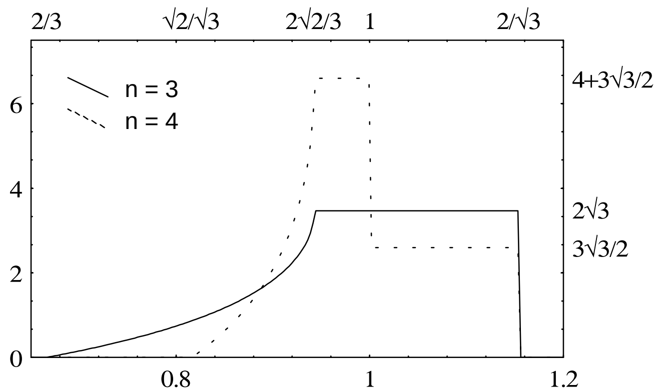


Fig. 3. *Distribution density of estimates of size of totally anisotropic object with systematic projections into regular sets of 3 and 4 directions in the space.*

Distribution densities of estimates of the size of a totally anisotropic object, by systematic projections, with the regular sets of directions (n = 1, 2, 3, 4, 6) can be calculated using spherical trigonometry. The distribution of the estimate with n = 1 is uniform, the graphs of those with directions of normals to cube and octahedron (n = 3, 4) are shown in Fig. 3, and those with n = 2 and 6 have shapes similar to the distributions with n = 3 and 4 respectively.

Coefficients of variation of estimates of the size of the objects with partial anisotropy by projections into regular sets of directions (Fig. 4) grow from 0 to values of CV of objects having either total anisotropy (6) or cylindrical anisotropy. As in the plane (3) the more equidistant directions of projections in the space are, the faster the variation coefficient tends to 0 if the ratio of the lengths of ellipsoid axes tends to 1:

$$CV_n \cong \delta_n \ln^{\nu_n} a/c \qquad a/c \to 1 \tag{8}$$

$\nu_1 = \nu_2 = 1,\ \nu_3 = \nu_4 = 2,\ \nu_6 = 3;\ \delta_1 = \dfrac{2}{3\sqrt{5}}, \delta_2 = \dfrac{1}{3\sqrt{5}}, \delta_3 = \dfrac{2}{15\sqrt{21}}, \delta_4 = \dfrac{4}{45\sqrt{21}}, \delta_6 = \dfrac{8}{105\sqrt{143}}$

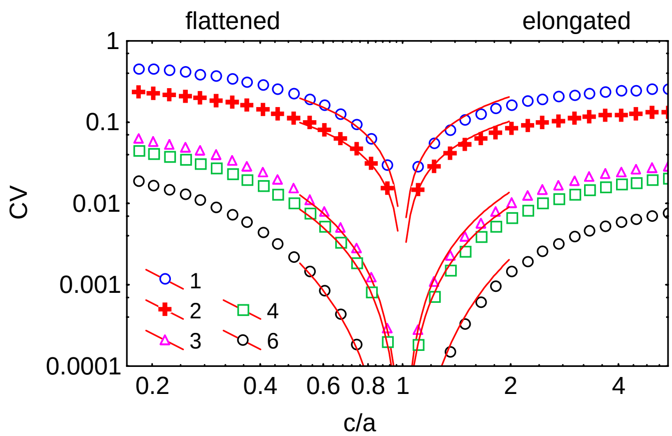


Fig. 4. *Variation coefficient of the estimate of surface area of ellipsoids with axes lengths a, a, c by projections into regular sets of directions in the space. Full line: asymptotic approximation (8).*

Weights w_i of the estimator (5) with minimal value of variation coefficient for a fixed set of directions and given object can be calculated using inverse $C^{-1} = D = (d_{ij})$ of covariance matrix $C = (c_{ij})$ of projections into the directions

$$w_i = \sum_j d_{ij} \Big/ \sum_{j,k} d_{jk} \tag{9}$$

3-D image analysis estimators of surface area (Meyer, 1992) use projections into 13 principal (orthogonal and diagonal) directions of a rectangular grid with the volume element dimensions equal to (dx, dy, dz). The estimators of length use projections into principal directions of the dual grid with volume element (dx^{-1}, dy^{-1}, dz^{-1}). If the grid is not cubic, the 13 principal directions are unevenly distributed and variation coefficient of the estimator may be even higher then that of the estimator with 3 orthogonal directions (Fig. 5). However, optimization of the weights (9) minimizing CV of anisotropic objects, using (4) and (6) for calculation of the covariance matrix, can also improve the estimates of the size of partially anisotropic objects (Fig. 6).

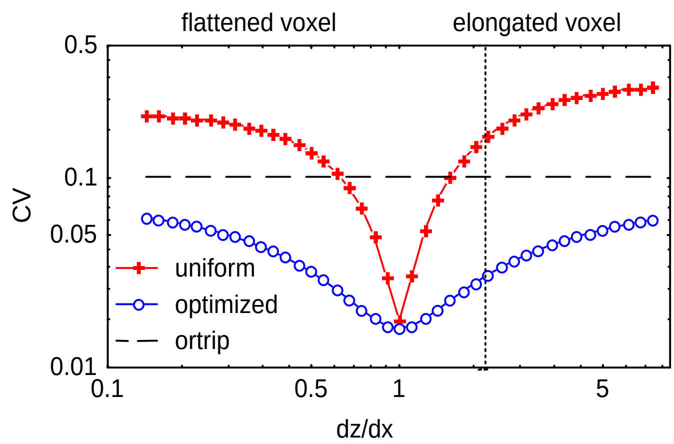


Fig. 5. *Coefficient of variation of the estimate of the size of totally anisotropic object by projections into 13 principal directions of orthogonal grid with volume element (voxel) dimensions dx:dx:dz.*

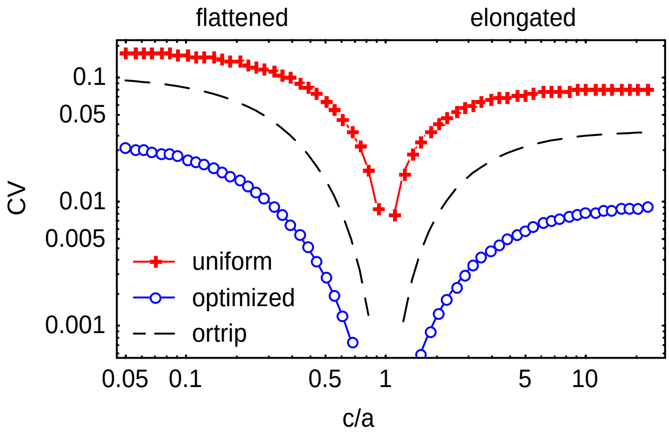
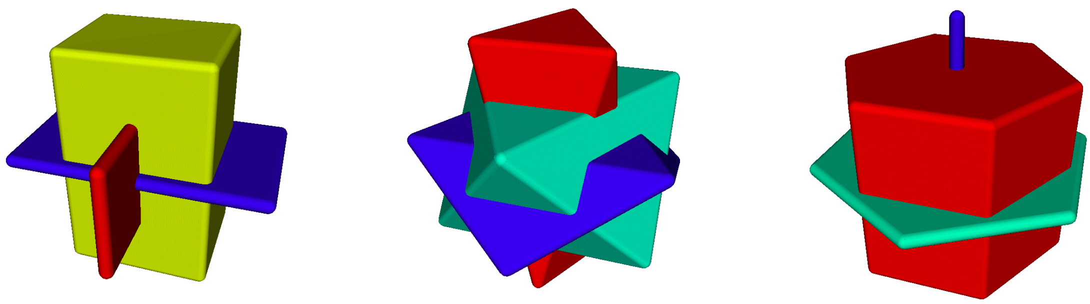


Fig. 6. *Variation coefficient of the estimate of the surface area of ellipsoids with axes lengths a, a, c by projections into 13 principal directions of orthogonal grid with volume element (voxel) dimensions 1:1:2.*

Sets of n = 2 to 13 directions in 3-D with minimal variation coefficient of the estimate of the size of a totally anisotropic object can be obtained by numerical minimisation of the directions and weights in (6) (Janáček, 1999). They are represented by the regular sets of directions for n = 2 to 4 and 6, the union of regular sets with n = 3 and 4 directions and with optimised coefficients (9) for n = 7, a pentagonal antiprism for n = 5, and sets with dihedral symmetry (D_2, D_3, D_6, D_5, D_4, D_2 respectively) for n = 8 to 13 (Fig. 7).



A B C

Fig. 7. *Graphical representation of sets of 8, 9 and 10 projection directions with minimal variation coefficient of the estimate of the size of a totally anisotropic object. The polyhedra vertices are the directions with the same weights; the colour hue is proportional to the relative value of weight (minima are red, maxima are blue).*

DISCUSSION

The above results on variation coefficient of systematic projections can help to compare efficiency of various stereological and image analysis estimators of length and surface area.

The suggested worst-case optimisation of image analysis estimators is easy to implement and provides an alternative to resampling of the image into a regular grid.

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REFERENCES

Janáček J (1999). Errors of spatial grids estimators of volume and surface area. Acta Stereologica 18(3):389-96.

Mattfeldt T, Möbius H-J, Mall G (1985). Orthogonal triplet probes: an efficient method for unbiased estimation of length and surface of objects with unknown orientation in space. J Microsc 139:279-89.

Meyer F (1992). Mathematical morphology: from two dimensions to three dimensions. J Microsc. 165:5-28.

Moran PAP (1966). Measuring the length of a curve. Biometrika 53:359-64.

Schuster EF (1974). Buffon's needle experiment. Am Math Month 81:26-9.

APPENDIX

The of estimate of the length of unit segment with direction φ by projections into n equidistant directions in the plane is:

$$g(\phi) = \pi/_{2n} \sum_{i=1}^{n} \left|\cos\left(\phi - i \cdot \pi/_{n}\right)\right|$$

Let $\beta = \pi/_{2n}$ $0 \le \phi \le \beta$ then if n is even: $g(\phi) = \frac{\beta \cos(\beta - \phi)}{\sin \beta}$ if n is odd: $g(\phi) = \frac{\beta \cos \phi}{\sin \beta}$