3.

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$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\cos^2\varphi d\theta d\varphi}{\sqrt{1-\sin^2\theta\sin^2\varphi}}=\frac{1}{2}\left(\frac{\Gamma\left(\frac{1}{4}\right)^4}{16\pi}-\frac{4\pi^3}{\Gamma\left(\frac{1}{4}\right)^4}\right)$$

Proof. By integration of series

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\cos^2\varphi d\theta d\varphi}{\sqrt{1-\sin^2\theta\sin^2\varphi}}=\frac{\pi^2}{8}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^3}{n!^2\,(n+1)!}$$

obviously

$$\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^3}{n!^2\,(n+1)!}=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^3}{n!^3}-\frac{1}{8}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n^3}{n!\,(n+1)!\,(n+2)!}$$

and we recognize in the first and second sum expressions with the hypergeometric functions ${}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;1\right)$ and $\frac{1}{2}{}_3F_2\left(\frac{3}{2},\frac{3}{2},\frac{3}{2};3,2;1\right)$ that can be calculated by Dixon identity and Watson identity, respectively. □

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