Approximate information capacity of the perfect integrate-and-fire neuron using the temporal code
(Supplementary material)

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Here we provide a derivation of Eq.(20) in the main manuscript, i.e., we prove that

$$C_V = \frac{\sqrt{\sigma^2}}{\mu} = \sqrt{\frac{a(\lambda+\omega)}{S(\lambda-\omega)}}, \tag{1}$$

for the ISIs generated by the PIF model, where μ is the mean and σ^2 is the variance of ISIs. The probability density of ISIs is

$$f(t;\lambda,\omega) = \frac{S}{\sqrt{2\pi(\lambda+\omega)a^2t^3}} \exp\left\{-\frac{[S+(\lambda-\omega)at]^2}{2(\lambda+\omega)a^2t}\right\}. \tag{2}$$

Since S and a are known, we re-parametrize Eq. (2) by employing parameters $\beta > 0$ and $\gamma > 0$ as (Tweedie, 1956)

$$\beta = \frac{(\lambda-\omega)^2a^2}{2S^2}, \qquad \gamma = \frac{S^2}{(\lambda+\omega)a^2}, \tag{3}$$

and obtain

$$f(t;\beta,\gamma) = \sqrt{\frac{\gamma}{2\pi t^3}} \exp\left[-\beta\gamma t + \gamma\sqrt{2\beta} - \frac{\gamma}{2t}\right]. \tag{4}$$

Direct calculation of moments for density in Eq. (4) leads to modified Bessel functions of the second kind. Tweedie (1956) uses a neat trick to avoid this by employing the cumulant generating function as follows. Calculate the logarithm of the Laplace transform of the density in Eq. (4), $g(s) = \ln \mathcal{L}[f(t;\beta,\gamma)](s)$, which is (almost) the cumulant generating function,

$$g(s) = \ln \int_0^\infty \sqrt{\frac{\gamma}{2\pi t^3}} \exp\left[-ts - \beta\gamma t + \gamma\sqrt{2\beta} - \frac{\gamma}{2t}\right] dt = \tag{5}$$

$$= \ln e^{\gamma\sqrt{2\beta}} \int_0^\infty \sqrt{\frac{\gamma}{2\pi t^3}} \exp\left[-(\beta+s/\gamma)\gamma t - \frac{\gamma}{2t}\right] dt. \tag{6}$$

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The integral in Eq. (6) is problematic, but its calculation can be avoided by noticing that substitution $\beta \leftarrow \beta + s/\gamma$ in Eq. (4) yields

$$f(t; \beta + s/\gamma, \gamma) = \sqrt{\frac{\gamma}{2\pi t^3}} \exp\left[-(\beta + s/\gamma)\gamma t + \gamma\sqrt{2\,(\beta + s/\gamma)} - \frac{\gamma}{2t}\right]. \tag{7}$$

Comparing Eq. (7) with the integrand in Eq. (6) gives (since $\gamma\sqrt{2\,(\beta + s/\gamma)}$ does not depend on t)

$$g(s) = \gamma\sqrt{2\beta} - \gamma\sqrt{2\left(\beta + \frac{s}{\gamma}\right)} + \ln\int_0^\infty f(t; \beta + s/\gamma, \gamma)\,dt = \tag{8}$$

$$= \gamma\sqrt{2\beta} - \gamma\sqrt{2\left(\beta + \frac{s}{\gamma}\right)}, \tag{9}$$

for all $s > -\gamma\beta$ due to positivity of the parameter β. The first two central moments are then

$$\mu = -\left.\frac{dg(s)}{ds}\right|_{s=0} = \frac{1}{\sqrt{2\beta}}, \tag{10}$$

$$\sigma^2 = \left.\frac{d^2g(s)}{ds^2}\right|_{s=0} = \frac{1}{2\gamma\sqrt{2\beta^3}}. \tag{11}$$

Thus,

$$\frac{1}{C_V^2} = \frac{\mu^2}{\sigma^2} = \gamma\sqrt{2\beta} = \frac{(\lambda - \omega)S}{(\lambda + \omega)a}, \tag{12}$$

and therefore Eq. (1) holds. Note, that Eq. (1) can be found, e.g., in Tuckwell (1988) (with details of the calculation omitted).

References

Tuckwell, H. C., 1988. Introduction to Theoretical Neurobiology, Volume 2. Cambridge University Press, New York.

Tweedie, M. C. K., 1956. Statistical Properties of Inverse Gaussian Distributions. I. Ann. Math. Stat. 28 (2), 362–377.